METRIC GEOMETRY, FALL 2018
EXERCISE 1, IDEAS FOR SOLUTIONS.

Exercise 1. If B is Borel, then B^c is also Borel. In addition, open sets U with $B \subset U$ and closed sets C with $C \subset B^c$ are in bijective correspondence by complementation. Moreover, if we have $C = U^c$ for such U, C, it follows that

$$\mu(B^c) - \mu(C) = (\mu(X) - \mu(B)) - (\mu(X) - \mu(U)) = \mu(U) - \mu(B).$$

It follows that inner regularity for B is equivalent in our situation with outer regularity for B^c. Note that the equivalence proof here already relies on the assumption $\mu(X) < \infty$.

Hence, as described in the exercise, it is enough to show that the set

$$\mathcal{F} = \{B \subset X \text{ Borel} : B, B^c \text{ inner regular}\}$$

has all Borel sets, and this is done by showing that it is a σ-algebra containing all closed sets of X. We note that it' clearl follows from the definition that $\mathcal{F}$ is closed under conjugation.

i): Let $B \subset X$ be closed. By a selection of $C = B$, it is obvious that B is inner regular. We pick a descending sequence of open sets $U_i, i \in \mathbb{Z}_+$ for which $B = \bigcap_i U_i$: for example $U_i = B_d(B, 1/i)$, the ball of radius $1/i$ around the set B. Since $\mu(X) < \infty$, we have $\mu(U_1) < \infty$, and therefore $\mu(B) = \lim_{i\to\infty} \mu(U_i)$. Therefore, B is outer regular, and by the previous considerations B^c is inner regular. It follows that $\mathcal{F}$ contains all closed sets of X.

ii) We're told to let $B_1, B_2 \in \mathcal{F}$ and select $C_i \subset B_i$ closed with $\mu(B_i \setminus C_i) < \varepsilon$. We can compute

$$\begin{aligned}
\mu((B_1 \cup B_2) \setminus (C_1 \cup C_2)) &\le \mu(B_1 \setminus (C_1 \cup C_2)) + \mu(B_2 \setminus (C_1 \cup C_2)) \\
&\le \mu(B_1 \setminus C_1) + \mu(B_2 \setminus C_2) \\
&< 2\varepsilon
\end{aligned}$$

and, since $(B_1 \cap B_2) \setminus (C_1 \cap C_2) \subset (B_1 \setminus C_1) \cup (B_2 \setminus C_2)$,

$$\begin{aligned}
\mu((B_1 \cap B_2) \setminus (C_1 \cap C_2)) &\le \mu(B_1 \setminus C_1) + \mu(B_2 \setminus C_2) \\
&< 2\varepsilon.
\end{aligned}$$

It follows that $B_1 \cup B_2$ and $B_1 \cap B_2$ are inner regular. Since $(B_1 \cup B_2)^c = B_1^c \cap B_2^c$ and $(B_1 \cap B_2)^c = B_1^c \cup B_2^c$, the same argument yields that their complements are inner regular. It follows that $\mathcal{F}$ is closed under finite unions and intersections.

iii) Suppose now that $B_1, B_2, \ldots \in \mathcal{F}$. We pick $D_1 = B_1$, $D_2 = B_2 \cap D_1^c$, $D_3 = B_3 \cap D_2^c$, and so on. By ii), we have $D_i \in \mathcal{F}$. It is also clear that D_i are disjoint and $\bigcup_i D_i = \bigcup_i B_i$.

iv) We now assume that $B = \bigcup_{i=1}^\infty B_i$ where $B_i \in \mathcal{F}$, and by iii) we may assume the B_i are disjoint. We then have $\sum_{i=1}^\infty B_i = \mu(B) < \infty$, so we find a $k \in \mathbb{Z}_+$ for which $\mu(B \setminus \bigcup_{i=1}^k B_i) = \sum_{i=k+1}^\infty \mu(B_i) < \varepsilon/2$. Furthermore, since

$\bigcup_{i=1}^k B_i \in \mathcal{F}$ as a finite union of $\mathcal{F}$-sets, we may pick a closed $C \subset \bigcup_{i=1}^k B_i$ for which $\mu(\bigcup_{i=1}^{k-1} B_i) - \mu(C) < \varepsilon/2$. Now $C \subset B$ and $\mu(B) - \mu(C) < \varepsilon$. Hence, countable unions of $\mathcal{F}$-sets are inner regular.

v) Assume that $B = \bigcap_{i=1}^\infty B_i$ where $B_i \in \mathcal{F}$. For every B_i, select a closed $C_i \subset B_i$ with $\mu(B_i \setminus C_i) < 2^{-i}\varepsilon$, and let $C = \bigcap_{i=1}^\infty C_i$. Note that $(\bigcap_{i=1}^\infty B_i) \setminus (\bigcap_{i=1}^\infty C_i) \subset \bigcup_{i=1}^\infty (B_i \setminus C_i)$. Hence, $\mu(B \setminus C) < \sum_{i=1}^\infty 2^{-i}\varepsilon = \varepsilon$. We obtain that countable intersections of $\mathcal{F}$-sets are inner regular.

vi) Since $(\bigcup_i B_i)^c = \bigcap_i B_i^c$, part v) implies that $(\bigcup_i B_i)^c$ is inner regular. Therefore, $\mathcal{F}$ is closed under countable unions. Since $\mathcal{F}$ contains all closed subsets of X and is closed under complementation and countable unions, it is a σ-algebra containing the Borel sets on X.

Exercise 2. Denote by X the whole domain of μ. Recall that for simplicity, we assumed that $\mu(X) < \infty$. We briefly remark in the start that in the definition

$$\mu'(S) = \inf\{\mu(B) : S \subset B \in \Sigma\}$$

the infimum is over a nonempty set, since Σ contains X which every $S \in \Sigma'$ is contained in.

i) Suppose $S \in \Sigma'$. By definition, we find $B_i \in \Sigma$ for which $S \subset B_i$ and $\mu'(S) \le \mu(B) \le \mu'(S) + i^{-1}$. We let $B = \bigcap_i B_i$, and note that $B \in \Sigma$. Then $S \subset B$ so $\mu'(S) \le \mu(B)$, but also $\mu(B) \le \mu(B_i) \le \mu'(S) + i^{-1}$ for all i. Hence, $\mu'(S) = \mu(B)$.

ii) We let Σ'' be the collection of sets $S' \subset X$ for which $C' \subset S' \subset B'$ for some $B', C' \in \Sigma$ with $\mu(B') = \mu(C')$. It is straightforward to see that Σ'' is closed under complements and countable unions, namely by considering complements and unions of the respective B' and C'. Also it's easily seen that $\Sigma \subset \Sigma''$ and $\mathcal{N} \subset \Sigma''$. It follows that $\Sigma' \subset \Sigma''$, and in particular that $S \in \Sigma''$.

Hence, we find the desirable $C \subset S \subset B$ with $\mu(C) = \mu(B)$, which by our simplifying assumption on finiteness of μ implies $\mu(B \setminus C) = 0$. Since $S \setminus C \subset B \setminus C$, it follows that $S \setminus C \in \mathcal{N}$, and we may select $N = S \setminus C$.

iii) The only nontrivial part in showing that μ' is a measure is to show that it has additivity. However, we first remark that μ is increasing, as it's of use in the proof. Note that when we select $C \subset S \subset B$ with $\mu(C) = \mu(B)$, this remains true if we shrink B as long as the shrunken down B still contains C. Hence, by i), we may assume $\mu'(S) = \mu(B) = \mu(C)$. Now, if $S \subset S'$, we pick corresponding B, C, B', C' and estimate $\mu'(S) = \mu(C) \le \mu(B') = \mu'(S')$.

Next we prove additivity. If $S_i \in \Sigma'$ are disjoint and we pick corresponding B_i, C_i, the sets C_i are also disjoint. By increasingness we get

$$\mu\Big(\bigcup_i S_i\Big) \le \mu\Big(\bigcup_i B_i\Big) = \mu\Big(\bigcup_i C_i\Big) \le \mu\Big(\bigcup_i S_i\Big).$$

Hence,

$$\mu\Big(\bigcup_i S_i\Big) = \mu\Big(\bigcup_i C_i\Big) = \sum_i \mu(C_i) = \sum_i \mu(S_i).$$

At this point, we now have that μ' is an extension of μ.

iv) Suppose then that $\tilde{\mu}$ extends μ to $\tilde{\Sigma} \supset \Sigma$, and $\tilde{\mu}$ is complete. We note that by completeness of $\tilde{\mu}$, we have $\mathcal{N} \subset \tilde{\Sigma}$, so therefore $\Sigma' \subset \tilde{\Sigma}$. Let

$S \in \Sigma'$. Then by ii), $S = C \cup N$ where $C \in \Sigma$ and $N \in \mathcal{N}$. Since N is contained in a nullset of Σ and $\tilde{\mu}$ extends μ, we have $\tilde{\mu}(N) = 0$, and therefore $\tilde{\mu}(S) = \mu(C) = \mu'(S)$, which completes the proof.

Exercise 3. It is well known from basic topology that compact implies complete. Moreover, since compactness of K implies that for a fixed $\varepsilon > 0$ the open cover $\{U(x, \varepsilon) : x \in K\}$ has a finite subcover (where U denotes the open ball as in exercise 11), we obtain that compact implies totally bounded.

For the converse, suppose K is complete and totally bounded, and let (x_j) be a sequence in K. We construct subsequences (x^i_j) inductively for $i = 1, 2, \dots$ as follows. Pick a finite cover of K by balls of radius $1/i$, select a ball B_{i+1} which has infinitely many points x^i_j, and let (x^{i+1}_j) be the subsequence of (x^i_j) contained in B_{i+1}. Then, let $(y_j) = (x^j_j)$ be the diagonal subsequence. It clearly follows that (y_j) is Cauchy, and therefore it has a limit due to completeness.

Exercise 4. By Exercise 1, we know that μ is inner regular by closed sets. Hence, it is enough to show that we have inner regularity by compact sets for all closed subsets of X.

Let $C \subset X$ be closed, and fix $\varepsilon > 0$. Separability means that we have a dense sequence (x_j) in X. We let $K^j_i = \bigcup_{k=1}^{j} B(x_k, 1/i) \cap C$ for $i, j \in \mathbb{Z}_+$. For each i, we have $\mu(C) = \lim_{j\to\infty} \mu(K^j_i)$, so we may select $K_i = K^{j_i}_i$ so that $j_1 < j_2 < \dots$ and $\mu(C \setminus K^{j_i}_i) < 2^{-i}\varepsilon$. Now let $K = \cap_{i=1}^{\infty} K_i$.

Since K is a closed subset of a complete space, it is complete. Furthermore, K admits finite covers by balls of radius $1/i$ for all $i \in \mathbb{Z}_+$, and therefore also finite covers by balls of radius $2/i$ centered at points of K. Hence, K is totally bounded, and therefore also compact. Moreover, we have $K \subset C$ and $\mu(C \setminus K) \leq \sum_{i=1}^{\infty} \mu(C \setminus K_i) < \varepsilon$. Hence, C is inner regular by compact sets.

The σ-finite case: Suppose (X, μ) is σ-finite, and select an increasing sequence X_i so that $X = \bigcup_i X_i$ and X_i have finite measure. We define the Borel measures μ_i on X by

$$\mu_i(A) = \mu(A \cap X_i)$$

(that is, μ_i is the restriction of the measure μ to the Borel set X_i). Now, for every i, (X, μ_i) is a complete separable metric measure space, and furthermore $\mu_i(X) = \mu(X_i) < \infty$.

Suppose that $S \subset X$ is a Borel set. Then, as the sequence X_i of sets is increasing, we have $\mu_i(S) \to \mu(S)$. We give here the case $\mu(S) < \infty$ (the other case $\mu(S) = \infty$ is similar). In this case, for every $\varepsilon > 0$, we can pick i so that $\mu_i(S) \geq \mu(S) - \varepsilon/2$. Applying the previously proven result on finite measures to (X, μ_i), we find a compact $K \subset S$ so that $\mu_i(K) \geq \mu_i(S) - \varepsilon/2$. But now

$$\mu(K) \geq \mu_i(K) \geq \mu_i(S) - \varepsilon/2 \geq \mu(S) - \varepsilon.$$

Hence, we may approximate S from inside by compact sets in μ.

Remark: It is possible that in the above σ-finite case, closed balls do not have finite measure. To see this, pick $X = \mathbb{R}$ with the standard Euclidean metric, and let μ be the counting measure on the rationals (ie. $\mu(A)$ is the number of rational numbers in A). Then μ is a Borel measure on a separable metric measure space, and $(\mathbb{R}, \mu)$ is σ-finite by $\{\mathbb{R} \setminus \mathbb{Q}, \{q_1\}, \{q_2\}, \ldots\}$. While this μ is inner regular by the above proof (also easy to see directly), it is neither outer regular nor does it have finite values on compact sets, and therefore it is not Radon.

Exercise 5. By separability, we have a dense sequence (x_j) in X. We build our $\mathcal{C}$ inductively as follows. Start from $\mathcal{C}_0 = \emptyset$. Then, assuming $\mathcal{C}_i$ is constructed, pick the x_j with the smallest index for which $x_j \in B \in \mathcal{B}$ for some B with $B \cap (\cup \mathcal{C}_i) = \emptyset$, and let $\mathcal{C}_{i+1} = \mathcal{C}_i \cup \{B\}$. Finally, whether this process terminates or continues for all natural number indices, set $\mathcal{C} = \bigcup_i \mathcal{C}_i$.

Clearly the resulting $\mathcal{C}$ is disjoint. Suppose then that $\mathcal{C}$ is not maximal: that is, there is a $B \in \mathcal{B}$ with $B \cap \cup \mathcal{C} = \emptyset$. By density of (x_j) we find a $x_k \in B$. However, now we obtain a contradiction, as $x_k \notin \cup \mathcal{C}$, but at least by step $i = k$, either B or some other ball of $\mathcal{B}$ containing x_k would have been added to $\mathcal{C}_i$.

Remark: The above proof requires the *axiom of countable choice* $\mathbf{AC}_\omega$: every countable collection of non-empty sets has a choice function. This, however, is weaker that the full axiom of choice $\mathbf{AC}$, which is equivalent to Zorn's lemma.

Exercise 6. A simple counterexample is provided by e.g. the family of all balls centered at zero in $\mathbb{R}^n$: any disjoint subcover cannot contain more than one ball B and the family contains balls larger than $5B$.

Exercise 7. As in the hint, let $B = B(x, r)$ be a ball, and use Exercise 5 to select a maximal disjoint subcollection $\mathcal{C}$ of balls of radius $r/4$ which are centered at points of B. If there is a $x \in B$ for which $x \notin 2B'$ for all $B' \in \mathcal{C}$, then $B(x, r/4) \cap B' = \emptyset$ for all $B' \in \mathcal{C}$, which contradicts maximality of $\mathcal{C}$. Hence, we obtain the desired B_i by inflating the balls of $\mathcal{C}$ by a factor of 2 if we can show a universal upper bound for the size of $\mathcal{C}$.

If $B' \in \mathcal{C}$, since B' is centered at a point of B and has radius $r/4$, we have $B \subset 8B'$. Hence, $\mu(B') \geq C^{-3}\mu(8B') \geq C^{-3}\mu(B)$, where C is the doubling constant. On the other hand, since $\mathcal{C}$ is disjoint and its sets are contained in $2B$, we have $\mu(B) \geq C^{-1}\mu(2B) \geq C^{-1} \sum_{B' \in \mathcal{C}} \mu(B')$. Hence, $\mathcal{C}$ cannot have more than C^4 elements, which is a global upper bound of the desired type.

Exercise 8. Let $B = B(x, r) \subset X$ be a closed ball. For every $i \in \mathbb{Z}_+$, we can apply metric doubling inductively to cover B with finitely many balls of radius $2^{-i}r$. Also, by the same radius-doubling argument as in Exercise 4, we may assume the balls have centerpoints in B, showing that B is totally bounded. Hence, by Exercise 3, completeness of X and closedness of B, we get that B is compact, and therefore X is proper.

For separability of X, fix a point x, let $i, j \in \mathbb{Z}_+$ take a ball $B(x, 2^j)$, and use metric doubling to cover it by finitely many balls of radius 2^{-i}. By doing this with the same x for every j, we obtain a countable cover of X by balls of radius 2^{-i}. By taking the centerpoints of these balls for all i, we obtain a countable subset S of X which is easily seen to be dense.

Exercise 9. Suppose $\mathcal{B}$ is a Vitali cover for an unbounded S. We take bounded $S_1 \subset S_2 \subset \dots$ so that $S = \bigcup_{i=1}^{\infty} S_i$. We construct the disjoint subcover $\mathcal{C} \subset \mathcal{B}$ in an inductive manner.

First, $\mathcal{B}$ is a Vitali cover of S_1, and by the bounded Vitali covering theorem we obtain a countable disjoint subset $\mathcal{C}_1'$ so that $\mu(S_1 \setminus \cup \mathcal{C}_1') = 0$. Since X is doubling and S_1 is bounded, S_1 has finite measure. Hence, we may pick a finite subset $\mathcal{C}_1'' \subset \mathcal{C}_1'$ so that $\mu(S_1 \setminus \cup \mathcal{C}_1'') < 2^{-1}$. Select $\mathcal{C}_1 = \mathcal{C}_1''$.

Now, let $\mathcal{B}_2$ be the collection of all $B \in \mathcal{B}$ which do not intersect $\cup \mathcal{C}_1$. Since $\mathcal{C}_1$ is finite, $\cup \mathcal{C}_1$ is closed, and therefore every point of $S_2 \setminus \cup \mathcal{C}_1$ has positive distance to $\cup \mathcal{C}_1$. Due to this, $\mathcal{B}_2$ is a Vitali cover of $S_2 \setminus \cup \mathcal{C}_1$. We may again use bounded Vitali to pick a countable disjoint subset $\mathcal{C}_2' \subset \mathcal{B}_2$ so that $\mu(S_2 \setminus \cup(\mathcal{C}_2' \cup \mathcal{C}_1)) = 0$, and then take a finite subset $\mathcal{C}_2'' \subset \mathcal{C}_2'$ so that $\mu(S_2 \setminus \cup(\mathcal{C}_2'' \cup \mathcal{C}_1)) < 2^{-2}$. Select $\mathcal{C}_2 = \mathcal{C}_2'' \cup \mathcal{C}_1$.

Continuing this inductively, we obtain $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \dots$ so that every $\mathcal{C}_i$ is disjoint and $\mu(S_i \setminus \cup \mathcal{C}_i) < 2^{-i}$. Set $\mathcal{C} = \bigcup_i \mathcal{C}_i$. We claim that $\mathcal{C}$ is the desired subset of $\mathcal{B}$ given by the Vitali covering theorem.

Disjointness of $\mathcal{C}$ is clear, so it remains to show that $\mu(S \setminus \cup \mathcal{C}) = 0$. Suppose to the contrary that $\mu(S \setminus \cup \mathcal{C}) > 0$. Then we find a S_i so that $\mu(S_i \setminus \cup \mathcal{C}) > 0$. But this is a contradiction, since $\mu(S_i \setminus \cup \mathcal{C})$ doesn't decrease as i increases, yet $\mu(S_i \setminus \cup \mathcal{C}) \leq \mu(S_i \setminus \cup \mathcal{C}_i) < 2^{-i}$ for all i.

Exercise 10. Consider the function $|f - q|$ for a $q \in \mathbb{Q}$. Since f is integrable, this function is locally integrable and non-negative, and therefore the weak Lebesgue differentiation theorem says that

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - q| \, \mathrm{d}\mu(y) = |f(x) - q|$$

for almost every $x \in X$. We let L_f be the set of points where the above holds for every $q \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable, we have $\mu(X \setminus L_f) = 0$.

Now, let $x \in L_f$, and select a $q \in \mathbb{Q}$ for which $|q - f(x)| < \varepsilon/2$. We obtain

$$\begin{aligned}
&\limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}\mu(y) \\
&\leq \limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} (|f(x) - q| + |f(y) - q|) \, \mathrm{d}\mu(y) \\
&= |f(x) - q| + \limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - q| \, \mathrm{d}\mu(y) \\
&= |f(x) - q| + |f(x) - q| \\
&< \varepsilon.
\end{aligned}$$

As this holds for arbitrary $\varepsilon > 0$, the claim follows.

Exercise 11. *i)* As described in the hint, a two-point space suffices: let $X = \{0, 1\}$ with the standard metric, and let μ be the counting measure. Every ball of X is nonempty, so the only possibilities are $\mu(B) = \mu(2B)$ and $2\mu(B) = \mu(2B)$, giving us a doubling constant of 2. Also clearly the set $B(0,1) \setminus U(0,1)$ has measure 1.

ii) We show how to make an atomless version of the previous example. Let d_∞ be the metric on $\mathbb{R}^2$ induces by the sup-norm. Recall that balls $B(x,r)$ in d_∞ are squares centered at x with width and height $2r$.

We select $X = \{0,1\} \times \mathbb{R} \subset \mathbb{R}^2$ and $d = d_\infty|_X$. Finally, we let μ be the measure on X given by adding the 1-dimensional Lebesgue measures on $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$. It follows that every $B(x,r)$ has μ-measure either $2r$ or $4r$, depending on whether r is less than 1 or not. Hence, μ is doubling with constant 4. Also, it's clear that $\mu(B((0,0),1) \setminus U((0,0),1)) = 2$, and that μ has no atoms.