EXERCISE SHEET 1

Throughout this exercise sheet, (X, d, μ) will be a metric measure space. Recall that this means that (X, d) is a complete metric space equipped with a σ-finite Borel measure μ.

Exercise 1. *Suppose that $\mu(X) < \infty$. In this exercise we will prove that, for any Borel $B \subset X$,*

$$\mu(B) = \sup\{\mu(C) : C \subset B \text{ closed}\} \tag{1}$$

and

$$\mu(B) = \inf\{\mu(U) : U \supset B \text{ open}\}. \tag{2}$$

That is, μ is inner regular *and* outer regular *respectively.*

By considering complements, it suffices to prove (1) *for all Borel $B \subset X$. To do this, let*

$$\mathcal{F} = \{B \subset X \text{ Borel} : B, B^c \text{ satisfy } (1)\}.$$

We will show that $\mathcal{F}$ is a σ-algebra that contains all closed sets, and hence all Borel sets.

i) *Let $C \subset X$ be closed. Note that C satisfies* (1)*. Prove that C satisfies* (2)*, and hence $B \in \mathcal{F}$.*
ii) *Let $B_1, B_2 \in \mathcal{F}$ and for $\epsilon > 0$ let $C_i \subset B_i$ be closed sets with $\mu(B_i \setminus C_i) < \epsilon$ for each $i = 1, 2$. Show that $\mu((B_1 \cap B_2) \setminus (C_1 \cap C_2)) < 2\epsilon$ and $\mu((B_1 \cup B_2) \setminus (C_1 \cup C_2)) < 2\epsilon$. Deduce that $B_1 \cap B_2$ and $B_1 \cup B_2$ belong to $\mathcal{F}$.*
iii) *By induction, this implies that $\mathcal{F}$ is closed under finite unions. We must show that it is closed under countable unions of sets. To this end, let $B_1, B_2, B_3, \ldots \in \mathcal{F}$. Prove that there exists pairwise disjoint $D_1, D_2, D_3, \ldots \in \mathcal{F}$ with*
$$\bigcup_{i=1}^{n} D_i = \bigcup_{i=1}^{n} B_i$$
for each $i \in \mathbb{N}$. In particular
$$\sum_{i \in \mathbb{N}} \mu(D_i) = \mu(B) < \infty.$$
iv) *Given $\epsilon > 0$ find a closed set $C \subset \cup_i B_i$ with $\mu(\cup_i B_i \setminus C) < \epsilon$, and hence that $\cup_i B_i$ satisfies* (1)*.*
v) *Given $\epsilon > 0$ find a closed set $C \subset \cap_i B_i$ with $\mu(\cap_i B_i \setminus C) < \epsilon$, and hence that $\cap_i B_i$ satisfies* (1)*.*
vi) *Deduce that $\cup_i B_i$ satisfies* (2) *and hence belongs to $\mathcal{F}$.*

Exercise 2. *A measure μ is* complete *if, whenever N is measurable with $\mu(N) = 0$ and $S \subset N$, S is also measurable. Suppose that μ is defined on a σ-algebra Σ, let*

$$\mathcal{N} = \{S \subset X : \exists N \in \Sigma,\ S \subset N,\ \mu(N) = 0\}$$

and let Σ' be the smallest σ-algebra containing $\Sigma \cup \mathcal{N}$. For simplicity, suppose that μ is finite. Define $\mu' : \Sigma' \to [0, \infty)$ by

$$\mu'(S) = \inf\{\mu(B) : S \subset B \in \Sigma\}.$$

i) *Prove that, for any $S \in \Sigma'$, there exists $B \in \Sigma$ with $S \subset B$ and $\mu'(S) = \mu(B)$.*

ii) *Hence deduce that, for any* $S \in \Sigma'$*, there exists* $B, C \in \Sigma$ *with* $C \subset S \subset B$ *and* $\mu(B \setminus C) = 0$ *and an* $N \in \mathcal{N}$ *such that* $S = C \cup N$*. Note that* (1) *and* (2) *hold for* μ'*.*

iii) *Prove that* μ' *is a measure.*

iv) *Now suppose that* $\tilde{\mu}$ *is another completion of* μ*. That is,* $\tilde{\mu}$ *is a complete measure defined on some* σ*-algebra* $\tilde{\Sigma} \supset \Sigma$ *and* $\tilde{\mu} = \mu$ *when restricted to* Σ*. Prove that* $\tilde{\Sigma} \supset \Sigma'$ *and for every* $S \in \Sigma'$

$$\tilde{\mu}(S) = \mu'(S).$$

Thus μ' *is the* smallest *completion of* μ*. (*Smallest *in the sense that any other completion of* μ *extends* μ'*)*

Exercise 3. *You are probably aware of two definitions of compactness: one defined by finite open covers and the other by convergent subsequences. For this exercise you can (and should) use either one of these definitions. In this exercise we will prove a third.*

A subset S *of a metric space* (X, d) *is* totally bounded *if, for any* $\epsilon > 0$*, there exist finitely many points* $x_1, x_2, \ldots, x_n \in K$ *such that*

$$S \subset B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \ldots \cup B(x_n, \epsilon).$$

Prove that $K \subset X$ *is compact if and only if it is complete and totally bounded.*

Remark: the Heine-Borel theorem states that a subset of $\mathbb{R}^n$ *is compact if and only if it is closed and bounded. This question proves the metric space generalisation of the Heine-Borel theorem.*

Exercise 4. *Suppose that* X *is separable and* $\mu(X) < \infty$*. Prove that* μ *is inner regular by compact sets:*

$$\mu(B) = \sup\{\mu(K) : K \subset B \text{ compact}\}.$$

Also prove that μ *is inner regular by compact sets if we only assume it is* σ*-finite.*

Exercise 5. *Suppose that* (X, d) *is separable and* $\mathcal{B}$ *is a collection of balls* $B \subset X$ *with positive radius. Prove (without using Zorn's lemma) that there exists a maximal disjoint sub-collection* $\mathcal{C} \subset \mathcal{B}$*. Here, "maximal" means that for any* $B \in \mathcal{B}$ *there exists a* $B' \in \mathcal{C}$ *with* $B \cap B' \neq \emptyset$*.*

Exercise 6. *Prove that the* $5r$ *covering lemma is false if we do not assume that the balls have uniformly bounded radii.*

Exercise 7. *A metric space* (X, d) *is* metric doubling *if there exists an* $N \in \mathbb{N}$ *such that, for any ball* $B \subset X$ *there exists balls* $B_1, B_2, \ldots, B_N \subset X$ *of half the radius of* B *such that*

$$B \subset B_1 \cup B_2 \cup \ldots \cup B_N.$$

Suppose that (X, d, μ) *is doubling. Prove that* (X, d) *is metric doubling. Hint: let* $B \subset X$ *be a ball and let* $\mathcal{C}$ *be a maximal disjoint collection of balls with one quarter the radius of* B *and prove a universal upper bound for the number of balls in* $\mathcal{C}$*.*

Exercise 8. *Prove that any complete metric doubling metric space is* proper*: closed balls are compact. Also prove that any metric doubling metric space is separable.*

Exercise 9. *Prove the Vitali covering theorem for the case when* S *is unbounded.*

Exercise 10. *Suppose that* (X, d, μ) *is doubling and* $f \colon X \to \mathbb{R}$ *is integrable. Improve the Lebesgue differentiation theorem by showing that*

$$\lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y) - f(x)| \; d\mu(y) = 0$$

for μ-a.e. $x \in X$. Hint: for $q \in \mathbb{Q}$ first apply the existing Lebesgue differentiation theorem to $f - q$.

Exercise 11. *Recall that usually we write $B(x, r)$ for the* closed *ball. For this question, we will write $U(x, r)$ for the open ball.*

i) *Write down a doubling metric measure space (X, d, μ) for which there exists $x \in X$ and $r > 0$ such that $B(x, r) \setminus U(x, r)$ has positive measure. Hint: such a space exists where X contains only two points.*
ii) *The previous example contains* atoms*: single points of positive measure. Construct a better example with no atoms. Hint: such a space exists where X is a subset of $\mathbb{R}^2$ equipped with the sup norm*

$$\|(x, y)\|_\infty = \max\{|x|, |y|\}.$$