EXERCISE SHEET 4

Exercise 1. *A previous exercise asked you to prove that a Poincaré inequality is preserved under bi-Lipschitz transformations. I cannot find an easy way to prove this. However, we do not need it for the proof of Cheeger's theorem. We only require the following.*

Let (X, d, μ) be a metric measure space for which there exists $C, \eta \geq 1$ such that, for any 1-Lipschitz $f : X \to \mathbb{R}$ and any upper gradient ρ of f,

$$\operatorname{Lip}(f, x)^\eta \leq C\rho(x)$$

for μ almost every $x \in X$.

Suppose that $\iota : X \to (Y, d')$ is bi-Lipschitz and $\nu = \iota_\# \mu$. Show that there exist $C', \eta' \geq 1$ such that, for any 1-Lipschitz $f : Y \to \mathbb{R}$ and any upper gradient ρ of f,

$$\operatorname{Lip}(f, y)^{\eta'} \leq C'\rho(y)$$

for ν almost every $y \in Y$.

Exercise 2. *Prove the converse to Proposition 6.2: Suppose that (U, ϕ) is a chart in a Lipschitz differentiability space. Show that there exists an $N \in \mathbb{N}$ such that, whenever $\psi : X \to \mathbb{R}^n$ is a Lipschitz function with*

$$\operatorname{Lip}(D \cdot \psi, x) > 0$$

for every $x \in U$ and every $D \in \mathbb{R}^n \setminus \{0\}$, then $n \leq N$.

Exercise 3. *For a metric space (X, d), let $\mathcal{C}$ be the collection of non-empty closed and bounded subsets of X. For $C, D \in \mathcal{C}$ define the* Hausdorff distance *between C and D to be*

$$d_H(C, D) = \inf\{r \geq 0 : C \subset B(D, r) \text{ and } D \subset B(C, r)\},$$

for $B(D, r)$ the r-neighbourhood of D:

$$B(D, r) = \bigcup_{d \in D} B(d, r).$$

i) *Prove that d_H is a metric on $\mathcal{C}$.*
ii) *Give an example to show that d_H may not be a metric on the set of non-empty bounded subsets of a metric space.*

Exercise 4. *Show that, if X is complete, then so is $\mathcal{C}$. Specifically, let C_n be a Cauchy sequence in $\mathcal{C}$. By taking a subsequence, we may suppose that $d(C_n, C_m) < 2^{-n}$ for all $m \geq n \in \mathbb{N}$. Define C to be the set*

$$C := \{c \in X : \forall\, n \in \mathbb{N}\ \exists\, c_n \in C_n,\ c_n \to c\}.$$

i) *Show that C is non-empty, bounded and closed (and hence belongs to $\mathcal{C}$).*
ii) *Given $\epsilon > 0$ show that there exists $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$, $C \subset B(C_n, \epsilon)$.*
iii) *For the other inclusion, for any $N \in \mathbb{N}$ and any $c^* \in C_N$ construct a Cauchy sequence $c_n \in C_n$ for which c_n converges to some point $c \in C$ with $d(c, c^*) \leq 2^{-N+1}$. In particular, $C_j \subset B(C, 2^{-N+1})$. Therefore, $C_n \to C$.*

Exercise 5. *Show that, if X is totally bounded, then so is $\mathcal{C}$. Hint: for any $\epsilon > 0$ let*

$$X \subset B(x_1, \epsilon) \cup \ldots \cup B(x_m, \epsilon)$$

and consider sets of the form

$$\bigcup_{i=1}^{j} B(y_i, \epsilon)$$

with $j \leq m$ and each $y_i \in \{x_1, \ldots, x_m\}$.

Therefore, if X is compact, so is $\mathcal{C}$. This is known as Blaschke's theorem.

Exercise 6. *There is another proof of the compactness statement. In fact, the compactness of $\mathcal{C}$ is equivalent to the Ascoli–Arzelà theorem.*

Suppose that X is compact and $C_n \in \mathcal{C}$.

i) *Prove that the functions $f_n(x) = d(x, C_n)$ are all 1-Lipschitz and there exists a closed interval $[0, d]$ into which each f_n is defined. Therefore, by the Ascoli–Arzelà theorem, (after possibly taking a subsequence) there exists a 1-Lipschitz $f \colon X \to \mathbb{R}_+$ such that $f_n \to f$.*
ii) *Show that*
$$C_n \to C := \{x \in X : f(x) = 0\}.$$

Now suppose that $f_n \colon X \to Y$ is a sequence of L-Lipschitz functions between two compact metric spaces. For each $n \in \mathbb{N}$ define C_n to be the graph *of f_n:*

$$C_n = \{(x, f(x)) \in X \times Y : x \in X\}.$$

iii) *Show that each C_n is a compact subset of $X \times Y$.*
iv) *If we assume that $\mathcal{C}(X \times Y)$ is compact, then (after passing to a subsequence) there exists a $C \in \mathcal{C}$ with $C_n \to \mathcal{C}$. Show that $\mathcal{C}$ is the graph of a function $f \colon X \to Y$.*
v) *Prove that $f_n \to f$ uniformly.*